Lecture 3: Covering and fibration

The goal of this lecture is to develop basic techniques to compute examples of fundamental groups through geometric covering. In particular, we will prove

$$\pi_1(S^1) = \mathbb{Z}.$$

Similar method applies to many other examples.

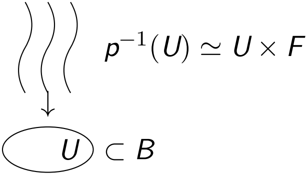
Fiber bundle and covering

Definition

Let $p : E \to B$ be in **Top**. A trivialization of p over an open set $U \subset B$ is a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ over U, i.e. , the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times F \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

p is called locally trivial if there exists an open cover $\mathcal{U}$ of B such that p has a trivialization over each open $U \in \mathcal{U}$. Such p is called a fiber bundle, F is called the fiber and B is called the base.



We denote it by

$$F \to E \to B$$

If we can find a trivialization of p over the whole B, then E is homeomorphic to $F \times B$

$$E \cong F \times B$$

and we say p is a trivial fiber bundle.

Example

The projection map

$$\mathbb{R}^{m+n} \to \mathbb{R}^n, \quad (x_1, \cdots, x_n, \cdots, x_{n+m}) \to (x_1, \cdots, x_n)$$

is a trivial fiber bundle with fiber $\mathbb{R}^m$.

Example

A real vector bundle of rank n over a manifold is a fiber bundle with fiber $\mathbb{R}^n$.

Example

We identity S^{2n+1} as the unite sphere in $\mathbb{C}^{n+1}$

$$S^{2n+1} = \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} | |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

There is a natural free S^1-action on S^{2n+1} given by

$$e^{i\theta} : (z_0, \cdots, z_n) \to (e^{i\theta} z_0, \cdots, e^{i\theta} z_n), \quad e^{i\theta} \in S^1.$$

The orbit space is the n-dim complex projective space $\mathbb{CP}^n$

$$S^{2n+1}/S^1 \cong \mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*.$$

Then the projection $S^{2n+1} \to \mathbb{CP}^n$ is a nontrivial fiber bundle with fiber S^1. The case when $n = 1$ gives the Hopf fibratioin

$$S^1 \to S^3 \to S^2.$$

Definition

A covering (space) (F-covering) is a locally trivial map $p : E \to B$ with discrete fiber F. A covering map which is a trivial fiber bundle is also called a trivial covering. If the fiber F has n points, we also call it a n-fold covering.

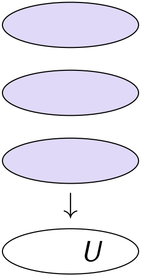


图: Local picture for a 3-fold covering

Example

The map $\exp : \mathbb{R}^1 \to S^1, \quad t \to e^{2\pi i t}$ is a $\mathbb{Z}$-covering.

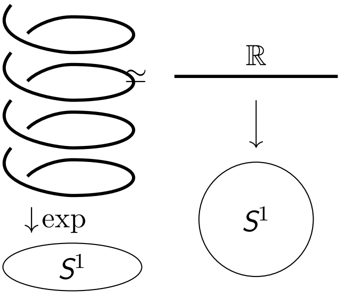


图: The $\mathbb{Z}$-covering of S^1

If $U = S^1 - \{-1\}$, then

$$\exp^{-1}(U) = \bigsqcup_{n\in\mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2}).$$

Example

Denote by $\mathbb{RP}^n$ the real projective space of dimension n, i.e.

$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/(\underline{x} \sim t\underline{x}), \quad \forall t \in \mathbb{R} - \{0\}, \underline{x} \in \mathbb{R}^{n+1} - \{0\}.$$

Let S^n be the n-sphere. Then there is a natural double cover

$$S^n \to \mathbb{RP}^n.$$

Example

The map $S^1 \to S^1$, $e^{2\pi \mathbf{i}\theta} \mapsto e^{2\pi \mathbf{i} n\theta}$ is $|n|$-fold covering, $n \in \mathbb{Z} - \{0\}$.

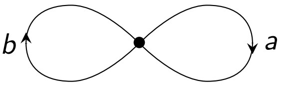
Example

The map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$, is not a covering (why?). But

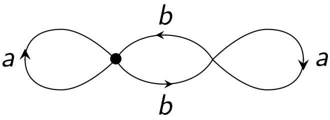
- the map $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$, is a $|n|$-covering, where $\mathbb{C}^* = \mathbb{C} - \{0\}$and $n \in \mathbb{Z} - \{0\}$.
- the map $\exp : \mathbb{C} \to \mathbb{C}^*$, $z \to e^{2\pi iz}$ is a $\mathbb{Z}$-covering.

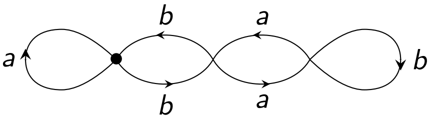
Example

The Figure-8



has two coverings as follows (the left is a 2-fold (or double) covering and the right is a 3-fold covering).





The 4-regular tree is a covering which is simply connected.

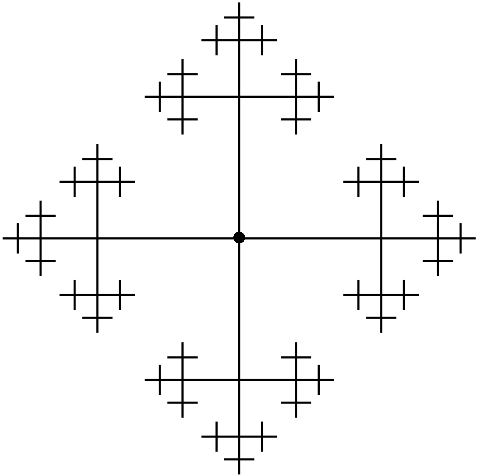


图: 4-regular tree

Example

Denote by $S_{g,b}$ the genus g surface with b boundary components.

- The surface $S_{4,0}$ admits a 7-fold covering from $S_{22,0}$.
- In general, $S_{g,b}$ admits a m-fold covering from $S_{mg-m+1,mb}$.

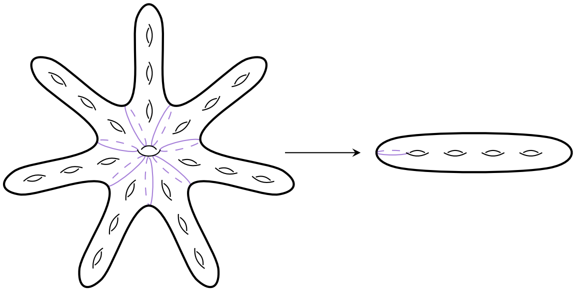
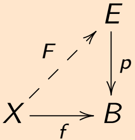


图: A 7-fold covering

Definition

Let $p : E \to B, f : X \to B$. A lifting of f along p is a map $F : X \to E$ such that $p \circ F = f$



Theorem (Uniqueness of lifting)

Let $p : E \to B$ be a covering. Let $F_0, F_1 : X \to E$ be two liftings of f. Suppose X is connected and F_0, F_1 agree somewhere. Then

$$F_0 = F_1.$$

We first state a simple lemma before proving the theorem.

Lemma

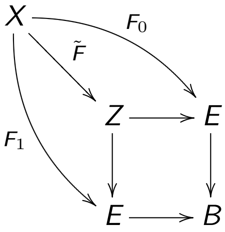
Let $p : E \to B$ be a covering. Let

$$D = \{(x, x) \in E \times E | x \in E\}$$
$$Z = \{(x, y) \in E \times E | p(x) = p(y)\}.$$

Then $D \subset Z$ is both open and closed.

Proof of Theorem

Let D, Z be defined in Lemma. Consider the map $\tilde{F} = (F_0, F_1) : X \to Z \subset E \times E$. By assumption, we have $\tilde{F}(X) \cap D \neq \emptyset$. Moreover, Lemma implies that $\tilde{F}^{-1}(D)$ is both open and closed. Since X is connected, we find $\tilde{F}^{-1}(D) = X$ which is equivalent to $F_0 = F_1$.

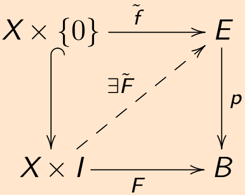


□

Hurwitz fibration

Definition

A map $p : E \to B$ is said to have the homotopy lifting property (HLP) with respect to X if for any maps $\tilde{f} : X \to E$ and $F : X \times I \to B$ such that $p \circ \tilde{f} = F|_{X\times 0}$, there exists a lifting $\tilde{F}$ of F along p such that $\tilde{F}|_{X\times 0} = \tilde{f}$.



Definition

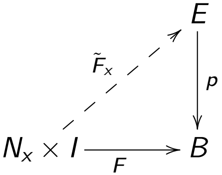
A map $p : E \to B$ is called a fibration (or Hurwitz fibration) if p has HLP for any space.

Theorem

A covering map is a fibration.

Proof

Let $p : E \to B, f : X \to B, \tilde{f} : X \to E, F : X \times I \to B$ be the data as in the definition of HLP. We only need to show the existence of $\tilde{F}_x$ for some neighbourhood N_x of any given point $x \in X$.



In fact, for any two such neighbourhoods N_x and N_y with $N_x \cap N_y \neq \emptyset$, we have $\tilde{F}_x|_{N_0}$ and $\tilde{F}_y|_{N_0}$ agree at some point by $\tilde{f}|_{N_0}$ and hence agree everywhere in $N_x \cap N_y$ by the uniqueness of lifting. Thus $\{\tilde{F}_x \mid x \in X\}$ glue to give the required lifting $\tilde{F}$.

Next, we proceed to prove the existence. Since I is compact, given $x \in X$ we can find a neighbourhood N_x and a partition

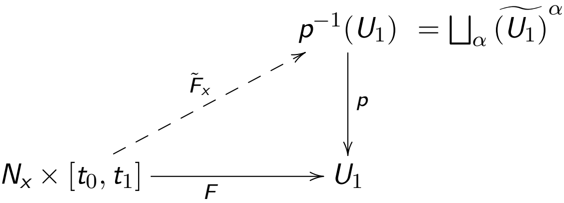
$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that p has a trivialization over open sets

$$U_i \supset F(N_x \times [t_i, t_{i+1}]).$$

Now we construct the lifting $\tilde{F}_x$ on $N_x \times [t_0, t_k]$, for $1 \leq k \leq m$, by induction on k.

- For $k = 1$, the lifting $\tilde{F}_x$ on $N_x \times [t_0, t_1]$ to one of the sheets of $p^{-1}(U_1)$ is determined by $\tilde{f}|_{N_x \times \{0\}}$:



- Assume that we have constructed $\tilde{F}_x$ on $N_x \times [t_0, t_k]$ for some k. Now, the lifting of $\tilde{F}_x$ on $N_x \times [t_k, t_{k+1}]$ to one of the sheets of $p^{-1}(U_k)$ is determined by $\tilde{f}|_{N_x \times \{t_k\}}$, which can be glued to the lifting on $N_x \times [t_0, t_k]$ by the uniqueness of lifting again. This finish the inductive step.

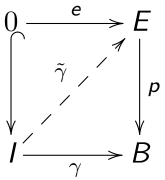
We obtain a lifting $\tilde{F}_x$ of F on $N_X \times I$ as required. $\square$

Corollary

Let $p : E \to B$ be a covering. Then for any path $\gamma : I \to B$ and $e \in E$ such that $p(e) = \gamma(0)$, there exists a unique path $\tilde{\gamma} : I \to E$ which lifts γ and $\tilde{\gamma}(0) = e$.

Proof.

Apply HLP to $X = \mathrm{pt}$.



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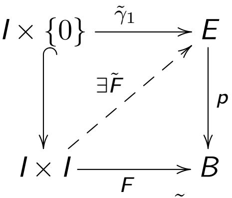
Corollary

Let $p : E \to B$ be a covering. Then $\Pi_1(E) \to \Pi_1(B)$ is a faithful functor. In particular, the map $\pi_1(E, e) \to \pi_1(B, p(e))$ is injective.

Proof.

Let $\tilde{\gamma}_i \colon I \to E$ be two paths and $[\tilde{\gamma}_i] \in \mathrm{Hom}_{\Pi_1(E)}(e_1, e_2)$. Let $\gamma_i = p \circ \tilde{\gamma}_i$. Suppose $[\gamma_1] = [\gamma_2]$ and we need to show that $[\tilde{\gamma}_1] = [\tilde{\gamma}_2]$. Let $F \colon \gamma_1 \simeq \gamma_2$ be a homotopy. Consider the following commutative diagram with the lifting $\tilde{F}$ by HLP



Then the uniqueness of lifting implies $\tilde{F}|_{I \times \{1\}} = \tilde{\gamma}_2$. Thus, $\tilde{\gamma}_1 \overset{\tilde{F}}{\simeq} \tilde{\gamma}_2$. □

Transport functor

Let $p : E \to B$ be a covering. Let $\gamma : I \to B$ be a path in B from b_1 to b_2. It defines a map

$$T_\gamma : p^{-1}(b_1) \to p^{-1}(b_2)$$
$$e_1 \to \tilde{\gamma}(1)$$

where $\tilde{\gamma}$ is a lift of γ with initial condition $\tilde{\gamma}(0) = e_1$.

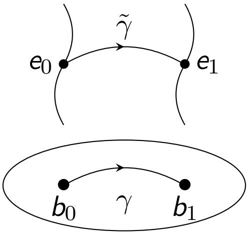


图: The trasportation

Assume $[\gamma_1] = [\gamma_2]$ in B. HLP implies that $T_{\gamma_1} = T_{\gamma_2}$. We find a well-defined map:

$$T : \mathrm{Hom}_{\Pi_1(B)}(b_1, b_2) \to \mathrm{Hom}_{\mathbf{Set}}(p^{-1}(b_1), p^{-1}(b_2))$$
$$[\gamma] \to T_{[\gamma]}$$

Definition

The following data

$$T : \Pi_1(B) \to \mathbf{Set}$$
$$b \to p^{-1}(b)$$
$$[\gamma] \to T_{[\gamma]}.$$

defines a functor, called the transport functor. In particular, we have a well-defined map

$$\pi_1(B, b) = \mathrm{Aut}_{\Pi_q(B)}(b) \to \mathrm{Aut}_{\mathbf{Set}}(p^{-1}(b)).$$

Example

Consider the covering map

$$\mathbb{Z} \to \mathbb{R}^1 \stackrel{\exp}{\to} S^1.$$

Consider the following path representing an element of $\pi_1(S^1)$

$$\gamma_n : I \to S^1, \quad t \to \exp(nt) = e^{2\pi int}, \quad n \in \mathbb{Z}.$$

Start with any point $m \in \mathbb{Z}$ in the fiber, γ_n lifts to a map to $\mathbb{R}^1$

$$\tilde{\gamma}_n : I \to R^1, \quad t \to m + nt.$$

We find $T_{[\gamma_n]}(m) = \tilde{\gamma}(1) = m + n$. Therefore $T_{[\gamma_n]} \in \mathrm{Aut}_{\mathbf{Set}}(\mathbb{Z})$ is

$$T_{[\gamma_n]} : \mathbb{Z} \to \mathbb{Z}, \quad m \to m + n.$$

Proposition

Let $p : E \to B$ be a covering, E be path connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E, e)$. In other words,

$$p^{-1}(b) \simeq \pi_1(B, b)/\pi_1(E, e)$$

as a coset space, i.e. we have the following short exact sequence

$$\begin{array}{c} 1 \to \pi_1(E, e) \to \pi_1(B, b) \xrightarrow{\partial_e} p^{-1}(b) \to 1. \\ \qquad\qquad\qquad\qquad [\gamma] \mapsto T_\gamma(e) \end{array}$$

Proof

For any point $e' \in p^{-1}(b)$, let $\tilde{\gamma}\colon e \to e'$ be a path in E and $\gamma = p \circ \tilde{\gamma}$. Then $e' = \partial_e(\gamma)$. This shows the surjectivity of ∂_e.

HLP implies that $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is injective and we can view $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$. By definition, for $\tilde{\gamma} \in \pi_1(E, e)$, we have $\partial_e([p \circ \tilde{\gamma}]) = \tilde{\gamma}(1) = e$, i.e. $\pi_1(E, e) \subset \mathrm{stab}_e(\pi_1(B, b))$. On the other hand, if $T_\gamma(e) = e$, then the lift $\tilde{\gamma}$ of γ is a loop, i.e. $\tilde{\gamma} \in \pi_1(E, e)$. Therefore, $\pi_1(E, e) \supset \mathrm{stab}_e(\pi_1(B, b))$. This implies

$$\pi_1(E, e) = \mathrm{stab}_e(\pi_1(B, b)).$$

□

Example

Consider the covering map

$$\mathbb{Z} \to \mathbb{R}^1 \xrightarrow{\exp} S^1.$$

Apply the previous proposition, we find an identification (as sets)

$$\deg : \pi_1(S^1) \simeq \mathbb{Z}.$$

This is called the degree map. An example of degree n map is

$$S^1 \to S^1, \quad e^{i\theta} \to e^{in\theta}.$$

The element $\gamma_n \in \pi_1(S^1)$ with $\deg(\gamma_n) = n$ acts on the fiber $\mathbb{Z}$ as

$$T_{\gamma_n} : \mathbb{Z} \to \mathbb{Z}$$
$$a \to a + n.$$

It is easy to see that

$$T_{\gamma_n} \circ T_{\gamma_m} = T_{\gamma_{n+m}}.$$

This implies that the degree map

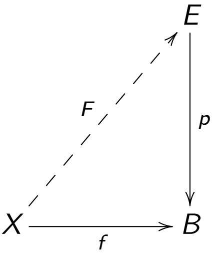
$$\deg : \pi_1(S^1) \to \mathbb{Z}$$

is a group isomorphism. Therefore $\pi_1(S^1) = \mathbb{Z}$.

Theorem (Lifting Criterion)

Let $p : E \to B$ be a covering. Let $f : X \to B$ where X is path connected and locally path connected. Let $e_0 \in E, x_0 \in X$ such that $f(x_0) = p(e_0)$. Then there exists a lift F of f with $F(x_0) = e_0$ if and only if

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0)).$$



$$\Longleftrightarrow f_*(\pi_1(X)) \subset \pi_1(E)$$

Proof

If such F exists, then

$$f_*(\pi_1(X, x_0)) = p_*\Big(F_*(\pi_1(X, x_0))\Big) \subset p_*(\pi_1(E, e_0)).$$

Conversely, let

$$\tilde{E} = \{(x, e) \in X \times E | f(x) = p(e)\} \subset X \times E$$

and consider the following commutative diagram

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B. \end{array}$$

The projection $\tilde{p}$ is also a covering. We have an induced commutative diagram of functors

$$\begin{array}{ccc} \Pi_1(X) & \longrightarrow & \Pi_1(B) \\ & {\scriptstyle T}\searrow & \downarrow {\scriptstyle T} \\ & & \mathbf{Set} \end{array}$$

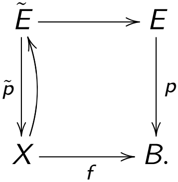
which induces natural group homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(B, b_0) \to \mathrm{Aut}(\tilde{p}^{-1}(x_0)) = \mathrm{Aut}(p^{-1}(b_0)).$$

Here $b_0 = f(x_0) = p(e)$. Let $\tilde{e}_0 = (x_0, e_0) \in \tilde{E}$. The condition $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$ says that $\pi_1(X, x_0)$ stabilizes $\tilde{e}_0$. By the previous Proposition, this implies an isomorphism

$$\tilde{p}_* : \pi_1(\tilde{E}, \tilde{e}_0) \simeq \pi_1(X, x_0).$$

Since X is locally path connected, $\tilde{E}$ is also locally path connected. Then path connected components and connected components of $\tilde{E}$ coincide. Let $\tilde{X}$ be the (path) connected component of $\tilde{E}$ containing $\tilde{e}$, then $\pi_1(\tilde{E}, \tilde{e}) \simeq \pi_1(X, x_0)$ implies that $\tilde{p} : \tilde{X} \to X$ is a covering with fiber a single point, hence a homeomorphism. Its inverse defines a continuous map $X \to \tilde{E}$ whose composition with $\tilde{E} \to E$ gives F.



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